Digital Holography in the Light of Phase Space

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Abstract— Digital holography is an imaging technique that has application in three dimensional imaging as well as phase contrast microscopy. In digital holography an interference pattern is recorded by a digital camera. The interference pattern is formed from a known reference eave and the light scattered from an object. The digital hologram is then input to a computer algorithm that simulates the back propagation of light from the camera plane to the image plane. In this paper we show how Phase Space Diagrams can be used to interpret and understand these systems.

1. INTRODUCTION

Holography is an imaging technique that was invented by Gabor in 1948 [1], for which he later received a Nobel prize. The technique was appended to remove the twin image by Leith and Upatnieks [2, 3]. Digital Holography is an optoelectronic version of this imaging technique [4–7]. The first half of digital holography is to record either one or multiple interferograms using a digital camera. These interferograms are recorded by intersecting a known reference beam with the light scattered by a reflecting or transmissive object. The second half of a digital holographic system is to numerically reconstruct the image using the recorded hologram as input to a numerical algorithm that simulates optical propagation of the wave field to the image plane. There are a host of different algorithms that can do this [8–12], each varying in accuracy and time taken and each providing a different point spread function for the overall record-reconstruction system.

Aside from applications in 3D imaging, one of the most important applications of digital holography is in the area of phase contrast microcopy [13, 14]. Microscopic digital holographic systems make use of a microscopic objective between the object and CCD and have become increasingly popular in recent years. This is firstly because because one can employ phase contrast techniques to the phase of the reconstructed image and secondly because it becomes possible to remove various types of aberrations digitally from these systems. In this paper we discuss both the recording and the reconstruction sides of digital holography from the phase space perspective. We show how phase space diagrams may be used to gain insight into the holographic process.

The Wigner Distribution function is a time frequency representation of signals which was introduced to the optics community My Bastians [15–17]. A simple graphical chart known as a 'Wigner chart' or 'Phase Space Diagram' (PSD) has often proved to be quite useful in understanding optical systems [18]. Indeed this approach has previously been employed to analyze holographic systems. In this paper we take the first steps to applying this approach to digital holographic systems by Lohmann and others [19–21]. The benefit here is that the digital camera can be classified as have a square like PSD — it has a physical width and a bandwidth defined by the inverse of the pixel pitch. By contrasting the PSD of the digital camera with that of the hologram to be recorded, we show how PSDs may be used to understand and interpret digital holography and to define optimum parameters.

2. WIGNER DISTRIBUTION FUNCTION BASICS

The Wigner Distribution function (WDF) is a bilinear transformation defined as follows.

$$W_u(x, f) = \int_{-\infty}^{\infty} u\left(x + \frac{x'}{2}\right) u^*\left(x - \frac{x'}{2}\right) \exp\left(-j2\pi f x'\right) dx' \quad (1)$$

The WDF is a function of both the spatial coordinate x and the spatial frequency coordinate f. It has a number of useful properties that make it particularly useful for studying propagation through optical systems. Below we list the properties of the WDF that are of interest in this paper.

1. The first property that is of interest in this paper is the coordinate mapping of the WDF that occurs when the signal of interest undergoes a Fresnel propagation, i.e., free space propagations as described under the paraxial approximation. A fresnel transformation can be described as follows, $u(x) \rightarrow u(x) * \exp\left(j\frac{\pi}{\lambda z}x^2\right)$ where the $*$ denotes a convolution, λ is the wavelength

of the light and z is the propagation distance. In this case the coordinates of the WDF are mapped as follows $W_u(x, f) \rightarrow W_u(x - f\lambda z, f)$

2. The second property of interest is the WDF of the product of two signals. In this case the WDF of the product may be described by the convolution of the WDFs of the individual signals along the f axis, i.e., the WDF of $u(x)v(x)$ is given by $W_{uv}(x, f) = W_u(x, f) *^f W_v(x, f)$. This property is of interest in this paper when we describe the intensity of a signal as the product of the signal with its conjugate.
3. This leads us to our third property. The WDF of the conjugate of signal is equivalent to the WDF of the signal inverted along f, i.e., $W_{u^*}(x, f) = W_u(x, -f)$
4. The fourth property concerns the marginals of the WDF. It can be shown that $\int_{-\infty}^{\infty} W_u(x, f)df = |u(x)|^2$ and similarly $\int_{-\infty}^{\infty} W_u(x, f)dx = |U(f)|^2$ where $U(f)$ is the Fourier Transform of $u(x)$. If a signal has the vast majority of its energy distributed in x over a support (or width) W and it also has the vast majority of its energy distributed over a support (or bandwidth) B, then we can say that WDF has similar properties. The energy will be contained in a region in (x, f) defined by W and B. This property leads to the formation of the Phase Space Diagram (PSD) which is at the heart of this paper.
5. The fifth and final property of the WDF concerns sampling, a topic which is always of interest when dealing with digital cameras. The Nyquist criterion states that if a signal is sampled with a sampling interval T, this is sufficient to completely recover a signal that has a finite bandwidth B so long as the following condition is satisfied: $T \leq \frac{1}{B}$ [22]. However it has recently been shown that a signal can be completely recovered so long as $T \leq \frac{1}{L}$ where L is the local bandwidth of the signal. For a full description of this subject in terms of the WDF, the reader may consult chapter 10 in [23]. This property of the WDF is of particular interest when we consider the sampling (digital hologram recording) of a Fresnel propagated signal. As we shall see, such a signal often has a local bandwidth L that reduces linearly with the distance of propagation z, while the total bandwidth B remains constant under propagation.

3. DIGITAL HOLOGRAM RECORDING

In Fig. 1, we illustrate a typical optical setup that is used to record digital holograms. Having been spatially filtered using a pinhole the beam is expanded and it now passes through a beam splitter. One arm is aimed directly at an object, or in the case of transmissive objects it is aimed through the object. The light coming from the object is incident on a CCD via a second beam splitter. The second arm is the reference beam which takes an approximately equidistant path to the CCD via the same latter two elements. There are numerous elements that are not shown in the diagram. For example a final linear polarizer in the setup serves to force both the reference and object wave fields into the same state of polarization such that the diffraction efficiency of the systems is maximized. In addition we use neutral density filters, half wave plate, linear polarizer and the polarizing beam splitters, so that we can effectively change the ratio of powers in the two arms of system. We write the interference pattern recorded by the camera as

$$I(x) = |u_z(x)|^2 + |r(x)|^2 + u_z(x)r^*(x) + u_z^*(x)r(x) \tag{2}$$

where $u_z(x)$ represents the signal that has propagated a distance z from the object to the object to the CCD (we recall that this is given by $u(x) * \exp\left(j\frac{\pi}{\lambda z}x^2\right)$) and $r(x)$ denote the reference wave field at the plane of the CCD. In our setup a phase shifting mirror is moved and four captures enabling separation of $u_z(x)$ [24]. This signal is then numerically reconstructed by simulating a backwards Fresnel propagation, and thus we obtain an image of $u(x)$.

4. DIGITAL HOLOGRAPHY DESCRIBED USING PHASE SPACE DIAGRAMS

We now demonstrate that the PSD is a very useful tool for investigating a PSI Digital holography recording system. We begin by showing the PSD of the original signal before any propagation has taken place. For a 3D object, we can imagine this signal to lie in the plane immediately after the object. This is the leftmost figure in Fig. 2. We can see that the object signal $u(x)$ has a physical width W_O and a bandwidth B_O and thus we can describe the signal as having a a square-like PSD. This follows from property 4 of the WDF discussed above. After the signal propagates a distance z (to the plane of the camera) the PSD now takes the form shown in the center part of Fig. 2. This is mathematically described by property 1 above. We can see that the signal spreads out in

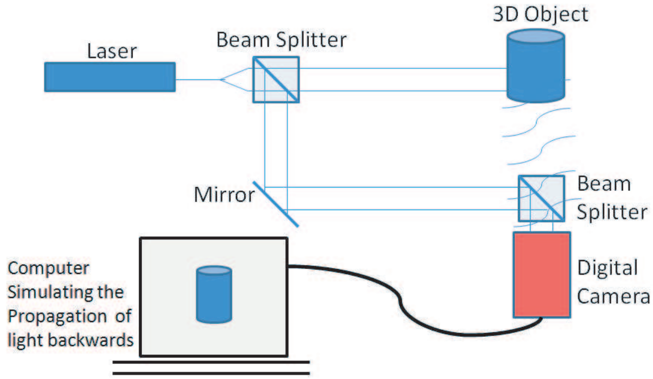

Figure 1: Digital holography in-line architecture. In our experimental set up we employ phase retarders (not shown) that allow us to make multiple captures with different constant phase shifts of the reference beam. We then implement the phase shifting interferometry algorithm to extract the complex wavefront.

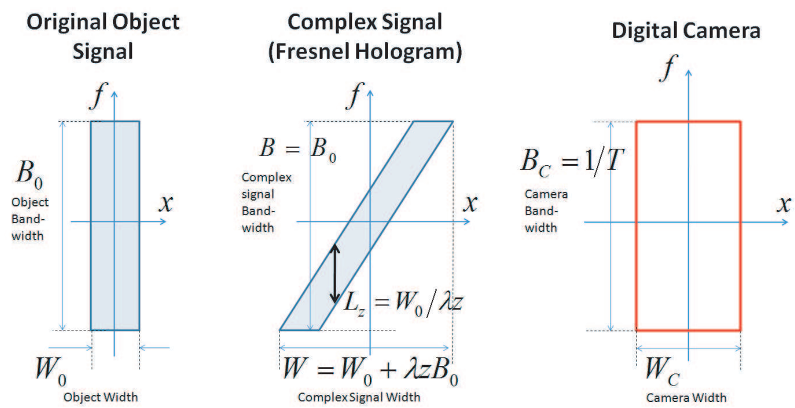

Figure 2: Phase space diagrams of original complex signal, the propagated complex signal in the plane of the camera and of the digital camera itself.

space and now occupies a larger support. We can deduce from simple geometry that the width of the signal will be given by $W_O + B_O\lambda z$ and the local bandwidth is given by $L = \frac{W_O}{\lambda z}$. We note that since the light diverges at all angles from the object, B_O will be infinite and thus the signal will theoretically be spread out over an infinite support after propagation. The propagated signal is now incident on a CCD where it sampled over the width of the camera, W_C. The bandwidth of the camera B_C, or more accurately we can describe this as the maximum bandwidth of the signal that the CCD can record, is given by $1/T$. Thus the camera has a square like PSD as shown in the rightmost part of 2. Because PSI allows us to isolate $u_z x$ from Equation (2) we can consider only this signal as being recorded by the CCD. The PSDs of the recorded complex signal, and its intensity are shown in Fig. 3. From property 5 above we can deduce that the local bandwidth of $u_z x$ must be less than or equal to B_C. We can express this condition as follows, $\frac{W_O}{\lambda z} \leq \frac{1}{T}$. This gives us a minimum condition for z, $z \geq \frac{TW_O}{\lambda}$. In the literature the intensity of the signal is predicted to have a bandwidth that is equal to twice the bandwidth of $u_z(x)$ [22]. However from property 3 above we can predict that this bandwidth will be much less than this value and will be given by twice the local frequency of $u_z(x)$. We believe that this point is interesting in the context of off axis digital holography where we must take into account the recording of this intensity term. After the signal is reconstructed (numerical propagation a distance $-z$), we can again find its PSD by applying the coordinate mapping defined in property 1 to the PSD of the recorded signal. The PSD of the reconstructed signal is shown in Fig. 4. We can see that the local bandwidth of the reconstructed signal is defined by the camera width and is given by $\frac{W_C}{\lambda z}$ while the total bandwidth of the signal is much larger and is given by $\frac{W_C+W_O}{\lambda z}$. The PSD of the intensity is also shown in the Figure and from property 3 above we can colclude that is will be equal to $\frac{2W_C}{\lambda z}$.

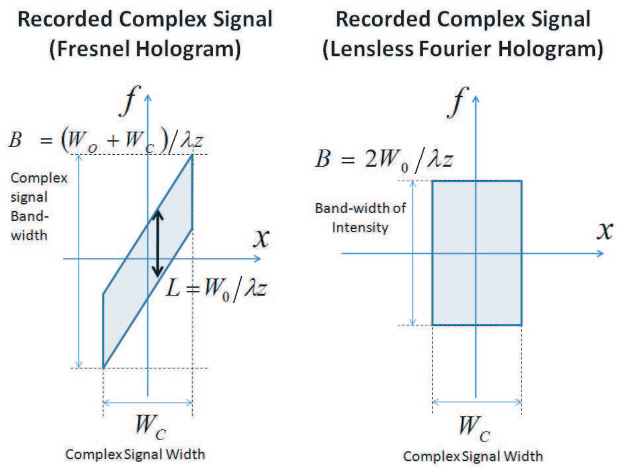

Figure 3: Phase space diagrams of the recorded complex signal and its intensity.

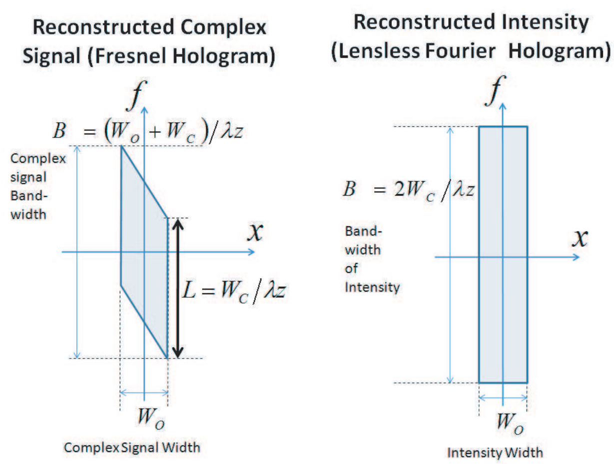

Figure 4: Phase space diagrams of the reconstructed complex signal and its intensity.

5. CONCLUSION

In this paper, we have analyzed digital holographic imaging using the properties of the Wigner Distribution function. Using Pase Space Diagrams we have found an expression for the minimum distance that we should place our object from the recording camera. We have also found expressions for the width, local bandwidth and overall bandwidth of the complex recorded and reconstructed signals, and their intensities intensity. We have demonstrated that the WDF is useful for understanding and interpreting digital holographic imaging.

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